

October 15, 2002

Path integral for relativistic oscillators: model of the Klein-Gordon particle in AdS space

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Abstract

Explicit path integration is carried out for the Green's functions of special relativistic harmonic oscillators in (1+1)- and (3+1)-dimensional Minkowski space-time modeled by a Klein-Gordon particle in the universal covering space-time of the anti-de Sitter static space-time. The energy spectrum together with the normalized wave functions are obtained. In the non-relativistic limit, the bound states of the one- and three-dimensional ordinary oscillators are regained.

PACS 03.65-Quantum theory, quantum mechanics.

I. Introduction

Relativistic problems that can be solved exactly by the use of the path integral approach are very limited especially for two reasons.

(1) For a relativistic particle with spin, the propagator cannot be described by a simple path integral based on any reasonable action. The fact that the spin has no classical origin makes it difficult to propose for it continuous paths¹.

(2) If the particles interact with each other or with an external potential, they can produce quantum effects which cannot be described by path fluctuations alone. These effects can be handled by perturbation theory in the framework of the quantum field theory².

However, in recent years, there have been a few successful examples where the difficulty which concerns the spin has been shaped. The Dirac propagator for a free particle³ has been derived in the framework of a model where the spin is classically described by internal variables. The path integral treatment of the Dirac-Coulomb problem⁴ and a Dirac electron in a one-dimensional Coulomb potential on the half-line and in presence of an external superstrong magnetic field⁵ have been obtained via the Biedenharn transformation⁶. The electron in presence of a constant magnetic field⁷ and the problem of charged particles in interaction with an electromagnetic plane wave alone⁸ or plus a parallel magnetic field⁹ have been studied by introducing a fifth parameter in order to bring the problem into a non-relativistic form. The relativistic spinless Coulomb system¹⁰ and the Klein-Gordon particle in vector plus scalar Hulthén-type potentials¹¹ have also been solved by path integration.

Recently, from various aspects (¹²⁻¹⁵ and references therein), there has been renewed interest in the relativistic harmonic oscillators. The problem of the special quantum relativistic oscillators represented by quantum free relativistic particles on the universal covering space-time of the anti-de Sitter static space-time (CAdS) has been studied through resolution of Klein-Gordon equation¹⁶. This model is constructed such that it recovers the ordinary harmonic oscillator in the non-relativistic limit.

There is, to our knowledge, no path integral discussion for the quantum relativistic harmonic oscillators. The purpose of the present paper is to fill this gap. The treatment will be restricted to spinless systems.

Our study is organized in the following way: in sec. 2, we construct the path integral associated with the $(1 + 1)$ -dimensional special relativistic

harmonic oscillator. The Green's function is derived in closed form, from which we obtain the energy spectrum and the normalized wave functions. In sec. 3, we extend the discussion to the $(3+1)$ -dimensional case. The radial Green's function is also given in closed form. The energy levels and the normalized wave functions are then deduced. The section 4 will be a conclusion.

II. The $(1+1)$ -dimensional special relativistic oscillator

The relativistic harmonic oscillator interaction in $(1+1)$ Minkowski space-time is equivalent to a free relativistic particle in the universal covering space-time of the anti-de Sitter space-time (CAdS). For a static form of the anti-de Sitter space-time metric, the line element is given by

$$ds^2 = \Lambda(x)c^2dt^2 - \frac{1}{\Lambda(x)}dx^2, \quad (1)$$

where

$$\Lambda(x) = 1 + \frac{\omega^2}{c^2}x^2. \quad (2)$$

Classical mechanics is described in this space by the classical Lagrangian and Hamiltonian, respectively:

$$L = -Mc\sqrt{1 - \frac{1}{\Lambda(x)}\frac{v^2}{c^2} + \frac{\omega^2}{c^2}x^2}, \quad (3)$$

$$H^2 = M^2c^4 + p^2c^2 + M^2\omega^2c^2x^2 + 2\omega^2x^2p^2 + \frac{\omega^4}{c^2}x^4p^2. \quad (4)$$

If we proceed by adopting the substitutions $H \rightarrow P_0 = i\hbar\frac{\partial}{\partial t}$, $p \rightarrow P_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$, $x \rightarrow x$, and in order to account for the ordering ambiguity of position- and momentum-operators, we introduce the parameters α and β by making the following replacements

$$\left\{ \begin{array}{l} x^4p^2 \rightarrow -\hbar^2 \left(x^4\frac{d^2}{dx^2} + 4x^3\frac{d}{dx} + \alpha x^2 \right), \\ x^2p^2 \rightarrow -\hbar^2 \left(x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + \beta \right), \end{array} \right. \quad (5)$$

the Green's function $G(x'', x')$ obeys the Klein-Gordon equation

$$\left\{ +\kappa\Lambda(x) + (1 - \alpha + 2\beta)\frac{\omega^4}{c^4}x^2 + \frac{\omega^2}{c^2} \right\} G(x'', x') = -\frac{1}{\hbar^2 c^2} \delta(x'' - x'). \quad (6)$$

where

$$= \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Lambda(x) \frac{\partial^2}{\partial x^2} \Lambda(x), \quad (7)$$

and

$$\kappa = \left(\frac{Mc}{\hbar} \right)^2 + (1 - 2\beta) \frac{\omega^2}{c^2}. \quad (8)$$

Note that choosing to work with the symmetrically ordered form (7) of the D'Alembertian operator, the quantization of the original problem will not be modified.

By using the Schwinger's integral representation¹⁷, the solution of the differential equation (6) can be written as follows:

$$G(x'', x') = \frac{1}{2i\hbar c^2} \int_0^\infty d\lambda \langle x'', t'' | \exp \left[\frac{i}{\hbar} H \lambda \right] | x', t' \rangle, \quad (9)$$

where the integrand $\langle x'', t'' | \exp \left[\frac{i}{\hbar} H \lambda \right] | x', t' \rangle$ is similar to the propagator of a quantum system evolving in λ time from (x', t') to (x'', t'') with the effective Hamiltonian,

$$H = \frac{1}{2} \left[-\Lambda(x) P_x^2 \Lambda(x) + \frac{P_0^2}{c^2} - \hbar^2 \kappa \Lambda(x) - \hbar^2 (1 - \alpha + 2\beta) \frac{\omega^4}{c^4} x^2 - \frac{\hbar^2 \omega^2}{c^2} \right]. \quad (10)$$

The integrand in Eq. (9) may be written as the path integral^{8,9,11,18,19}

$$\begin{aligned} P(x'', t'', x', t'; \lambda) &= \langle x'', t'' | \exp \left[\frac{i}{\hbar} H \lambda \right] | x', t' \rangle \\ &= \lim_{N \rightarrow \infty} \int \prod_{n=1}^N dx_n dt_n \prod_{n=1}^{N+1} \frac{d(P_x)_n}{2\pi\hbar} \frac{d(P_0)_n}{2\pi\hbar} \exp \left\{ \frac{i}{\hbar} \sum_{n=1}^{N+1} A_1^\varepsilon \right\}, \end{aligned} \quad (11)$$

with the short-time action

$$A_1^\varepsilon = (P_0)_n \triangle t_n - (P_x)_n \triangle x_n + \frac{\varepsilon}{2} \left(\frac{(P_0)_n^2}{c^2} - \Lambda(x_n) \Lambda(x_{n-1}) (P_x)_n^2 - \hbar^2 \frac{\omega^2}{c^2} - \hbar^2 \kappa \Lambda(x_n) - \hbar^2 (1 - \alpha + 2\beta) \frac{\omega^4}{c^4} x_n^2 \right), \quad (12)$$

where

$$\varepsilon = \frac{\lambda}{N+1} = ds, \quad (13)$$

and $s \in [0, \lambda]$ is a new time-like variable.

Let us first notice that the integrations on the variables t_n give N Dirac distributions $\delta((P_0)_n - (P_0)_{n+1})$. Thereafter the integrations on $(P_0)_n$ give $(P_0)_1 = (P_0)_2 = \dots = (P_0)_{N+1} = E$. The propagator (11) then becomes

$$P(x'', t'', x', t'; \lambda) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi\hbar} \exp \left[-\frac{i}{\hbar} E(t'' - t') \right] P_E(x'', x'; \lambda), \quad (14)$$

where the kernel $P_E(x'', x'; \lambda)$ is given by

$$P_E(x'', x'; \lambda) = \lim_{N \rightarrow \infty} \int \prod_{n=1}^N dx_n \prod_{n=1}^{N+1} \frac{d(P_x)_n}{2\pi\hbar} \exp \left[\frac{i}{\hbar} \sum_{n=1}^{N+1} A_2^\varepsilon \right], \quad (15)$$

with the short-time action

$$A_2^\varepsilon = -(P_x)_n \triangle x_n + \frac{\varepsilon}{2} \left[-\Lambda(x_n) \Lambda(x_{n-1}) (P_x)_n^2 + \frac{\hbar^2 \omega^2}{c^2} \left(\frac{E^2}{\hbar^2 \omega^2} - 1 \right) - \hbar^2 \kappa \Lambda(x_n) - \hbar^2 (1 - \alpha + 2\beta) \frac{\omega^4}{c^4} x_n^2 \right]. \quad (16)$$

Note that (14) is invariant under the change $E \rightarrow -E$.

Then, by integrating with respect to the variables $(P_x)_n$, we get

$$P_E(x'', x'; \lambda) = \lim_{N \rightarrow \infty} \int \prod_{n=1}^{N+1} \left[\frac{1}{2i\pi\hbar\varepsilon\Lambda(x_n)\Lambda(x_{n-1})} \right]^{\frac{1}{2}} \prod_{n=1}^N dx_n \exp \left[\frac{i}{\hbar} \sum_{n=1}^{N+1} A_3^\varepsilon \right], \quad (17)$$

with the short-time action in configuration space

$$A_3^\varepsilon = \frac{\Delta x_n^2}{2\varepsilon\Lambda(x_n)\Lambda(x_{n-1})} + \frac{\varepsilon}{2}\hbar^2 \left[\left(\frac{E^2}{\hbar^2\omega^2} - 1 \right) \frac{\omega^2}{c^2} - \kappa\Lambda(x_n) - (1 - \alpha + 2\beta) \frac{\omega^4}{c^4} x_n^2 \right]. \quad (18)$$

Substituting (14) into (9), we can rewrite (9) in the form:

$$G(x'', x') = \int_{-\infty}^{+\infty} \frac{dE}{2\pi\hbar} \exp \left[-\frac{i}{\hbar} E(t'' - t') \right] G_E(x'', x'), \quad (19)$$

with

$$G_E(x'', x') = \frac{1}{2i\hbar c^2} \int_0^\infty d\lambda P_E(x'', x'; \lambda). \quad (20)$$

If we next introduce a new variable u_n together with a rescaling of time²⁰ from ε to σ_n given by

$$\begin{cases} x_n = \frac{c}{\omega} \sinh u_n, \\ \varepsilon = \sigma_n \frac{c^2}{\omega^2} \frac{1}{\cosh u_n \cosh u_{n-1}}, \end{cases} \quad (21)$$

and incorporate the constraint

$$\lambda = \frac{c^2}{\omega^2} \int_0^S \frac{ds}{\cosh^2 u}, \quad (22)$$

by using the identity

$$\frac{c^2/\omega^2}{\cosh u'' \cosh u'} \int_0^\infty dS \delta \left(\lambda - \frac{c^2}{\omega^2} \int_0^S \frac{ds}{\cosh^2 u} \right) = 1, \quad (23)$$

the path integral (20) can be written as

$$G_E(x'', x') = \frac{1}{2i\hbar\omega c (\cosh u'' \cosh u')^{\frac{3}{2}}} \int_0^\infty dSP(u'', u'; S), \quad (24)$$

where

$$\begin{aligned}
P(u'', u'; S) = & \lim_{N \rightarrow \infty} \int \prod_{n=1}^{N+1} \left[\frac{1}{2i\pi\hbar\sigma_n} \right]^{\frac{1}{2}} \prod_{n=1}^N du_n \exp \left\{ \frac{i}{\hbar} \sum_{n=1}^{N+1} \left[\frac{\Delta u_n^2}{2\sigma_n} \right. \right. \\
& + \frac{\Delta u_n^4}{8\sigma_n} \left(\frac{1}{3} - \frac{1}{\cosh^2 \tilde{u}_n} \right) - \frac{\hbar^2}{2} \left(\frac{c^2}{\omega^2} \kappa - \frac{E^2/\hbar^2 \omega^2 - 1}{\cosh^2 \tilde{u}_n} \right. \\
& \left. \left. + (1 - \alpha + 2\beta) \tanh^2 \tilde{u}_n \right) \sigma_n \right] \Big\}. \quad (25)
\end{aligned}$$

Here we have used the usual abbreviations $\Delta u_n = u_n - u_{n-1}$, $\tilde{u}_n = \frac{u_n + u_{n-1}}{2}$, $u' = u(0)$ and $u'' = u(S)$. Note that the term in $(\Delta u_n)^4$ contributes significantly to the path integral. It can be estimated using the formula²²

$$\int_{-\infty}^{+\infty} \exp(-\alpha_1 x^2 + \alpha_2 x^4) dx = \int_{-\infty}^{+\infty} \exp \left(-\alpha_1 x^2 + \frac{3\alpha_2}{4\alpha_1^2} \right) dx, \quad (26)$$

valid for $|\alpha_1|$ large and $Re(\alpha_1) \succ 0$. This leads to

$$\begin{aligned}
P(u'', u'; S) = & \int Du(s) \exp \left\{ \frac{i}{\hbar} \int_0^S \left[\frac{\dot{u}^2}{2} - \frac{\hbar^2}{2} \left(\frac{c^2}{\omega^2} \kappa + \frac{1}{4} \right) \right. \right. \\
& \left. \left. + \frac{\hbar^2}{2} \frac{E^2/\hbar^2 \omega^2 - 1/4}{\cosh^2 u} - \frac{\hbar^2}{2} (1 - \alpha + 2\beta) \tanh^2 u \right] ds \right\}. \quad (27)
\end{aligned}$$

By noting that $\tanh^2 u = 1 - \frac{1}{\cosh^2 u}$, this last path integral is identical in form with that of the symmetric Rosen-Morse potential²¹ which has been studied recently²³⁻²⁷, but in order to obtain the equivalent to the Klein-Gordon equation in the AdS space-time we impose a restriction on the parameters α and β defined by the following two equations:

$$1 - \alpha + 2\beta = 0, \quad (28)$$

$$(1 - 2\beta) \frac{\omega^2}{c^2} = \xi R, \quad (29)$$

where $R = -2\frac{\omega^2}{c^2}$ is the scalar curvature and ξ is a numerical factor. Whence it follows that

$$\alpha = 2\xi + 2 \quad \text{and} \quad \beta = \xi + \frac{1}{2}. \quad (30)$$

In this case, the propagator (27) reduces to

$$P(u'', u'; S) = \int Du(s) \exp \left\{ \frac{i}{\hbar} \int_0^S \left[\frac{\dot{u}^2}{2} - \frac{\hbar^2}{2} \left(\left(\frac{Mc^2}{\hbar\omega} \right)^2 - 2\xi + \frac{1}{4} \right) + \frac{\hbar^2 E^2 / \hbar^2 \omega^2 - 1/4}{\cosh^2 u} \right] ds \right\}, \quad (31)$$

which is likewise the propagator relative to a symmetric Rosen-Morse potential. The Green's function with this potential has been evaluated through various techniques of path integration²³⁻²⁷, which reads in this case,

$$\begin{aligned} G(u'', u'; E) &= \int_0^{+\infty} dSP(u'', u'; S) \\ &= -\frac{i}{\hbar} \Gamma(\gamma - l_E) \Gamma(1 + l_E + \gamma) P_{l_E}^{-\gamma}(\tanh u'') P_{l_E}^{-\gamma}(-\tanh u'), \end{aligned} \quad (32)$$

where $P_{l_E}^{-\gamma}(\tanh u)$ is the associated Legendre function with

$$l_E = -\frac{1}{2} + \frac{E}{\hbar\omega}, \quad (33)$$

and

$$\gamma = \pm \frac{1}{2} \sqrt{1 + 4N^2 - 8\xi}, \quad N = \frac{Mc^2}{\hbar\omega}. \quad (34)$$

If we take into account Eqs. (30), insert (32) into (24), and remember the first equation of the transformation (21), we obtain the Green's function for the one-dimensional special relativistic harmonic oscillator under consideration

$$\begin{aligned} G_E(x'', x') &= -\frac{\Gamma(\gamma - l_E) \Gamma(1 + l_E + \gamma)}{2\hbar^2 \omega c} \left[\left(1 + \frac{\omega^2}{c^2} x''^2 \right) \left(1 + \frac{\omega^2}{c^2} x'^2 \right) \right]^{-\frac{3}{4}} \\ &\times P_{l_E}^{-\gamma} \left(\frac{\frac{\omega}{c} x''}{\sqrt{1 + \frac{\omega^2}{c^2} x''^2}} \right) P_{l_E}^{-\gamma} \left(-\frac{\frac{\omega}{c} x'}{\sqrt{1 + \frac{\omega^2}{c^2} x'^2}} \right). \end{aligned} \quad (35)$$

The poles of the Green's function yield the discrete energy spectrum. These are just the poles of $\Gamma(\gamma - l_E)$ which occur when $\gamma - l_E = -n$ for $n = 0, 1, 2, \dots$. They are given through the equations

$$\frac{1}{2} - \frac{E}{\hbar\omega} \pm \frac{1}{2} \sqrt{1 + 4N^2 - 8\xi} = -n. \quad (36)$$

So, algebraically we obtain two distinct sets of energy levels according to the positive and negative signs of the parameter γ . But we have to check whether the corresponding wave functions, which will be expressed in terms of the Legendre functions of the first kind $P_{l_E}^{-\gamma}(y)$ satisfy the boundary conditions for $y = \frac{\omega}{c}x / \sqrt{1 + \frac{\omega^2}{c^2}x^2} \rightarrow \pm 1$. By inspecting their asymptotic behaviors²⁸

$$P_{l_E}^{-\gamma}(y) \underset{y \rightarrow 1}{\simeq} \frac{(1-y)^{\frac{\gamma}{2}}}{2^{\frac{\gamma}{2}} \Gamma(1+\gamma)}; \quad \gamma \neq 0, -1, -2, -3, \dots, \quad (37)$$

$$P_{l_E}^{-\gamma}(y) \underset{y \rightarrow -1}{\simeq} \begin{cases} -\frac{\Gamma(-\gamma)}{2^{\frac{\gamma}{2}} \pi} \sin(l_E \pi) (1+y)^{\frac{\gamma}{2}} & \text{for } \text{Re}(\gamma) < 0, \\ \frac{2^{\frac{\gamma}{2}} \Gamma(\gamma)}{\Gamma(1+l_E+\gamma) \Gamma(\gamma-l_E)} (1+y)^{-\frac{\gamma}{2}} & \text{for } \text{Re}(\gamma) > 0, \end{cases} \quad (38)$$

we see that $P_{l_E}^{-\gamma}(y)$ diverges if $\text{Re}(\gamma) < 0$. Therefore we must choose the positive sign of γ and hence the energy eigenvalues are

$$E_n = \left(n + \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4N^2 - 8\xi} \right) \hbar\omega. \quad (39)$$

On the other hand the reality of the parameter γ implies the following range of the numerical factor $\xi < \frac{1}{8} (1 + 4N^2)$.

In the limit $c \rightarrow \infty$, the energy spectrum approaches

$$E_n^{NR} + Mc^2 = \hbar\omega \left(n + \frac{1}{2} \right) + Mc^2. \quad (40)$$

The first term gives the energy levels in the non-relativistic case and the second term is the rest energy of the harmonic oscillator.

The corresponding energy eigenfunctions can be found by approximation near the poles $\gamma - l_E \approx -n$:

$$\Gamma(\gamma - l_E) \approx \frac{(-1)^n}{n!} \frac{1}{\gamma - l_E + n}$$

$$= \frac{(-1)^{n+1} 2 \left(n + \gamma + \frac{1}{2}\right) \hbar^2 \omega^2}{n! (E^2 - E_n^2)}. \quad (41)$$

Using this behavior and the known property of the symmetry of the associated Legendre functions under spatial reflection, $x \rightarrow -x$, we get the contribution of the bound states to the spectral representation of the Green's function as

$$\begin{aligned} G_E(x'', x') &= \sum_{n=0}^{\infty} \frac{\omega}{c} \frac{n + \gamma + \frac{1}{2}}{n! (E^2 - E_n^2)} \Gamma(2\gamma + n + 1) \left(1 + \frac{\omega^2}{c^2} x''^2\right)^{-\frac{3}{4}} \\ &\quad \times \left(1 + \frac{\omega^2}{c^2} x'^2\right)^{-\frac{3}{4}} P_{n+\gamma}^{-\gamma} \left(\frac{\frac{\omega}{c} x''}{\sqrt{1 + \frac{\omega^2}{c^2} x''^2}}\right) P_{n+\gamma}^{-\gamma} \left(\frac{\frac{\omega}{c} x'}{\sqrt{1 + \frac{\omega^2}{c^2} x'^2}}\right) \\ &= \sum_{n=0}^{\infty} \frac{\Psi_n^\gamma(x'') \Psi_n^{\gamma*}(x')}{E^2 - E_n^2}. \end{aligned} \quad (42)$$

The properly normalized wave functions are thus

$$\Psi_n^\gamma(x) = \left[\frac{\omega}{c} \frac{n + \gamma + \frac{1}{2}}{n!} \Gamma(2\gamma + n + 1) \right]^{\frac{1}{2}} \left(1 + \frac{\omega^2}{c^2} x^2\right)^{-\frac{3}{4}} P_{n+\gamma}^{-\gamma} \left(\frac{\frac{\omega}{c} x}{\sqrt{1 + \frac{\omega^2}{c^2} x^2}}\right). \quad (43)$$

Taking into account the relation between the Gegenbauer polynomials and the associated Legendre functions (see formula (8.936) p. 1031 in Ref.²⁹)

$$C_n^\lambda(t) = \frac{\Gamma(2\lambda + n) \Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda) \Gamma(n + 1)} \left[\frac{1}{4} (t^2 - 1) \right]^{\frac{1}{4} - \frac{\lambda}{2}} P_{\lambda+n+\frac{1}{2}}^{\frac{1}{2}-\lambda}(t), \quad (44)$$

and using the doubling formula (see Eq. (8.335.1), p. 938 in Ref.²⁹)

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right), \quad (45)$$

we can also express (43) in the form

$$\Psi_n^\gamma(x) = \left[\frac{\omega}{c} \frac{(\gamma + n + \frac{1}{2}) n!}{\Gamma(2\gamma + n + 1)} \right]^{\frac{1}{2}} (2i)^\gamma \Gamma\left(\gamma + \frac{1}{2}\right) \left(1 + \frac{\omega^2}{c^2} x^2\right)^{-\frac{1}{2}(\gamma + \frac{3}{2})}$$

$$\times C_n^{\gamma+\frac{1}{2}} \left(\frac{\frac{\omega}{c}x}{\sqrt{1+\frac{\omega^2}{c^2}x^2}} \right) \quad (46)$$

In the limit $c \rightarrow \infty$, $\gamma \rightarrow N = \frac{Mc^2}{\hbar\omega}$ and with the help of the formula (see Eq. (8.328.1), p. 937 in Ref.²⁹)

$$\lim_{z \rightarrow \infty} \frac{\Gamma(z+a)}{\Gamma(z)} e^{-a \ln z} = 1, \quad (47)$$

we see that

$$\begin{aligned} & \lim_{c \rightarrow \infty} \gamma^{\frac{n}{2}} \left[\frac{\omega}{c} \frac{(\gamma + n + \frac{1}{2}) n!}{\Gamma(2\gamma + n + 1)} \right]^{\frac{1}{2}} (2)^\gamma \Gamma\left(\gamma + \frac{1}{2}\right) \\ &= \lim_{c \rightarrow \infty} \left[\frac{\omega}{c\sqrt{\pi}} \frac{n!}{2^n} \right]^{\frac{1}{2}} \left[\frac{\Gamma(\gamma + \frac{1}{2})}{\Gamma(\gamma)} \right]^{\frac{1}{2}} = \left(\frac{M\omega}{\pi\hbar} \right)^{\frac{1}{4}} \sqrt{\frac{n!}{2^n}}. \end{aligned} \quad (48)$$

By the use of the limit relation (see Eq. (8.936.5), p. 1031 in Ref.²⁹)

$$\lim_{\lambda \rightarrow \infty} \lambda^{-\frac{n}{2}} C_n^{\frac{\lambda}{2}} \left(t \sqrt{\frac{2}{\lambda}} \right) = \frac{2^{-\frac{n}{2}}}{n!} H_n(t), \quad (49)$$

the wave functions of the harmonic oscillator in the non-relativistic approximation are naturally regained

$$\lim_{c \rightarrow \infty} \Psi_n^\gamma(x) = \left(\frac{M\omega}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} e^{-\frac{M\omega}{2\hbar}x^2} H_n \left(\sqrt{\frac{M\omega}{\hbar}} x \right), \quad (50)$$

where $H_n \left(\sqrt{\frac{M\omega}{\hbar}} x \right)$ is the Hermite polynomial of n th order.

III. The (3+1)-dimensional special relativistic oscillator

The special relativistic harmonic oscillator in $(3+1)$ Minkowski space-time is simulated in the universal covering space-time (CAdS) of the anti- de

Sitter space-time with a negative curvature $R = -12\frac{\omega^2}{c^2}$ and a static metric of the form

$$ds^2 = \Lambda(r)c^2 dt^2 - \frac{1}{\Lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (51)$$

where

$$\Lambda(r) = 1 + \frac{\omega^2}{c^2} r^2 \quad (52)$$

is chosen in order to impose the non-relativistic limit.

The Lagrangian reads

$$L = -Mc \sqrt{\Lambda(r) - \frac{v^2}{c^2} + \frac{\omega^2 (\vec{r} \cdot \vec{v})^2}{c^4 \Lambda(r)}} \quad (53)$$

and the classical Hamiltonian is given by

$$H^2 = \Lambda(r) \left(M^2 c^4 + p^2 c^2 + \omega^2 (\vec{r} \cdot \vec{p})^2 \right). \quad (54)$$

As in the one-dimensional relativistic oscillator to construct the quantum counterpart of (54), we must respect the ordering ambiguity of the position- and momentum-operators. For this purpose, we introduce the parameters α, β, η , and ρ by making the following substitutions

$$\left\{ \begin{array}{l} x_i^4 p_i^2 \rightarrow -\hbar^2 \left(x_i^4 \frac{\partial^2}{\partial x_i^2} + 4x_i^3 \frac{\partial}{\partial x_i} + \alpha x_i^2 \right), \\ x_i^2 p_i^2 \rightarrow -\hbar^2 \left(x_i^2 \frac{\partial^2}{\partial x_i^2} + 2x_i \frac{\partial}{\partial x_i} + \beta \right), \\ x_i^3 p_i \rightarrow -i\hbar \left(x_i^3 \frac{\partial}{\partial x_i} + \eta x_i^2 \right), \\ x_i p_i \rightarrow -i\hbar \left(x_i \frac{\partial}{\partial x_i} + \rho \right). \end{array} \right. \quad (55)$$

In addition, since the spatial part of the $(3+1)$ AdS space-time has the spherical symmetry, we have to take $\eta = \frac{3}{2}$ and $\rho = \frac{1}{2}$. Then, the Green's function $G(\vec{r}'', t'', \vec{r}', t')$ satisfies the Klein-Gordon equation

$$(+U(r)) G(\vec{r}'', t''; \vec{r}', t') = -\frac{1}{\hbar^2 c^2} \delta(\vec{r}'' - \vec{r}') \delta(t'' - t'). \quad (56)$$

where

$$= \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Lambda(r) \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \Lambda(r) + \Lambda(r) \frac{\hat{l}^2}{\hbar^2 r^2}, \quad (57)$$

(\hat{l}^2 is the square of the orbital angular momentum operator) and $U(r)$ is the central potential

$$U(r) = \frac{\hbar^2 \omega^2}{c^2} \left[4\beta - \alpha - \frac{M^2 c^4}{\hbar^2 \omega^2} - 8 + \left(\alpha + 2\beta + \frac{7}{2} \right) \Lambda(r) \right]. \quad (58)$$

It follows that the Green's function $G(\vec{r}'', t'', \vec{r}', t')$ can be expanded into partial waves³¹ in spherical polar coordinates

$$G(\vec{r}'', t'', \vec{r}', t') = \frac{1}{r'' r'} \sum_{l=0}^{\infty} G_l(r'', t'', r', t') Y_l^{m*}(\theta'', \phi'') Y_l^m(\theta', \phi'), \quad (59)$$

where the radial Green's function, expressed in the Schwinger's integral representation¹⁷, is

$$G_l(r'', t'', r', t') = \frac{1}{2i\hbar c^2} \int_0^\infty d\lambda \langle r'', t'' | \exp \left[\frac{i}{\hbar} H_l \lambda \right] | r', t' \rangle, \quad (60)$$

The integrand in Eq. (60) is similar to the propagator of an harmonic oscillator which evolves in the time-like parameter λ with the effective Hamiltonian

$$H_l = \frac{1}{2} \left[-\Lambda(r) P_r^2 \Lambda(r) + \frac{P_0^2}{c^2} - \hbar^2 l(l+1) \frac{\Lambda(r)}{r^2} + U(r) \right]. \quad (61)$$

To find the energy eigenvalues $E_{n_r, l}$ and the wave functions $\Psi_{n_r, l}(r) = r^{-1} \Phi_{n_r, l}(r)$, we may evaluate (60) by path integration. The effective Hamiltonian (61) involves a singular centrifugal barrier at $r = 0$, so that the discrete form of the expression (60) is not defined due to path collapse. To obtain a tractable and stable path integral, we introduce an appropriate regulating function (following Kleinert³⁰) and write (60) in the form:

$$G_l(r'', t'', r', t') = \frac{1}{2i\hbar c^2} \int_0^\infty dS P_l(r'', t'', r', t'; S), \quad (62)$$

where the transformed path integral is given in the canonical form by

$$\begin{aligned}
P_l(r'', t'', r', t'; S) &= f_R(r'') f_L(r') \langle r'', t'' | \exp \left[\frac{i}{\hbar} S f_L(r) H_l f_R(r) \right] | r', t' \rangle, \\
&= f_R(r'') f_L(r') \int Dr(s) Dt(s) \int \frac{DP_r(s) DP_0(s)}{(2\pi\hbar)^2} \\
&\quad \times \exp \left\{ \frac{i}{\hbar} \int_0^S ds \left[-P_r \dot{r} + P_0 \dot{t} + f_L(r) H_l f_R(r) \right] \right\} \\
&= f_R(r'') f_L(r') \lim_{N \rightarrow \infty} \prod_{n=1}^N \left[\int dr_n dt_n \right] \\
&\quad \times \prod_{n=1}^{N+1} \left[\int \frac{d(P_r)_n d(P_0)_n}{(2\pi\hbar)^2} \right] \exp \left\{ \frac{i}{\hbar} \sum_{n=1}^{N+1} A_1^{\varepsilon_s} \right\}, \quad (63)
\end{aligned}$$

with the short-time action

$$\begin{aligned}
A_1^{\varepsilon_s} &= -(P_r)_n \triangle r_n + (P_0)_n \triangle t_n + \frac{\varepsilon_s}{2} f_L(r_n) \left[-\Lambda(r_n) \Lambda(r_{n-1}) (P_r)_n^2 + \frac{(P_0)_n^2}{c^2} \right. \\
&\quad \left. - \hbar^2 l(l+1) \frac{\Lambda(r_n)}{r_n^2} + U(r_n) \right] f_R(r_{n-1}), \quad (64)
\end{aligned}$$

and

$$\varepsilon_s = \frac{S}{N+1} = ds = \frac{d\tau}{f_L(r_n) f_R(r_{n-1})}; \quad d\tau = \varepsilon_\tau = \frac{\lambda}{N+1}. \quad (65)$$

The regulating function is defined as³⁰

$$f(r) = f_L(r) f_R(r) = f^{1-\lambda'}(r) f^{\lambda'}(r). \quad (66)$$

As in the $(1+1)$ -dimensional case, by doing successively the t_n and $(P_0)_n$ integrations we arrive at

$$P_l(r'', t'', r', t'; S) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dE \exp \left[-\frac{i}{\hbar} E(t'' - t') \right] P_l(r'', r'; S), \quad (67)$$

where the invariant kernel $P_l(r'', r'; S)$ under the change $E \rightarrow -E$ is given by

$$P_l(r'', r'; S) = f_R(r'') f_L(r') \lim_{N \rightarrow \infty} \prod_{n=1}^N \left[\int dr_n \right] \prod_{n=1}^{N+1} \left[\int \frac{d(P_r)_n}{(2\pi\hbar)} \right] \exp \left\{ \frac{i}{\hbar} \sum_{n=1}^{N+1} A_2^{\varepsilon_s} \right\}, \quad (68)$$

with

$$A_2^{\varepsilon_s} = -(P_r)_n \Delta r_n + \frac{\varepsilon_s}{2} f_L(r_n) \left[-\Lambda(r_n) \Lambda(r_{n-1}) (P_r)_n^2 + \frac{E^2}{c^2} - \hbar^2 l(l+1) \frac{\Lambda(r_n)}{r_n^2} + U(r_n) \right] f_R(r_{n-1}). \quad (69)$$

Substituting (67) into (62), we observe that the t-dependent term does not contain the variable S . Therefore, we can rewrite the partial Green's function (62) in the form:

$$G_l(r'', t'', r', t') = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dE \exp \left[-\frac{i}{\hbar} E(t'' - t') \right] G_l(r'', r'), \quad (70)$$

with

$$G_l(r'', r') = \frac{1}{2i\hbar c^2} \int_0^\infty dS P_l(r'', r'; S). \quad (71)$$

The path integration of the kernel $P_l(r'', r'; S)$ can be performed for any splitting parameter λ' . However, to simplify the calculation, we prefer to work with the mid-point prescription by taking $\lambda' = \frac{1}{2}$. This can be justified by the fact that the final result is independent of this parameter. Then, by integrating with respect to the variables $(P_r)_n$, we find

$$P_l(r'', r'; S) = \frac{[f(r') f(r'')]^{\frac{1}{4}}}{\Lambda(r') \Lambda(r'')} \lim_{N \rightarrow \infty} \prod_{n=1}^{N+1} \sqrt{\frac{1}{2i\pi\hbar\varepsilon_s}} \prod_{n=1}^N \left[\int \frac{dr_n}{\Lambda(r_n) \sqrt{f(r_n)}} \right] \times \exp \left\{ \frac{i}{\hbar} \sum_{n=1}^{N+1} A_3^{\varepsilon_s} \right\} \quad (72)$$

with the short-time action in configuration space

$$A_3^{\varepsilon_s} = \frac{\Delta r_n^2}{2\varepsilon_s \Lambda(r_n) \Lambda(r_{n-1}) \sqrt{f(r_n) f(r_{n-1})}} + \frac{\varepsilon_s}{2} f(r_n) \left[\frac{E^2}{c^2} - \hbar^2 l(l+1) \frac{\Lambda(r_n)}{r_n^2} + U(r_n) \right]. \quad (73)$$

We now use the following space transformation: $r \rightarrow u$, $r \in [0, \infty[$, $u \in]-\infty, \infty[$ defined by

$$r = \frac{c}{\omega} e^u. \quad (74)$$

The appropriate regulating function is then defined by

$$f(r(u)) = \frac{c^2}{4\omega^2 \cosh^2 u}. \quad (75)$$

By taking into account all the quantum corrections arising, of course, from the transformations (74) and (75), the Green's function (71) can straightforward be written as follows

$$G_l(r'', r') = \frac{1}{4i\hbar\omega c \sqrt{\Lambda(r'') \Lambda(r')} \cosh u'' \cosh u'} \int_0^\infty dS P_l(u'', u'; S), \quad (76)$$

with

$$P_l(u'', u'; S) = \int Du(s) \exp \left\{ \frac{i}{\hbar} \int_0^S \left[\frac{\dot{u}^2}{2} - \frac{\hbar^2}{4} (\nu^2 + k^2 + (\nu^2 - k^2) \tanh u) + \frac{\hbar^2}{8} \frac{\frac{E^2}{\hbar^2 \omega^2} + 4\beta - \alpha - 2}{\cosh^2 u} \right] \right\}, \quad (77)$$

where $\nu^2 = N^2 - 2\beta - \alpha + \frac{11}{4}$, $k = l + \frac{1}{2}$ and $N = Mc^2/\hbar\omega$.

This kernel is formally identical with that of the general Rosen-Morse (or general modified Pöschl-Teller) potential studied recently²³⁻²⁶. The Green's function associated to this potential is

$$G(u'', u'; E_{RM'}) = \int_0^\infty dS P_l(u'', u'; S). \quad (78)$$

As is shown by Kleinert³⁰, the Green's function of the general Rosen-Morse potential is related to the fixed-energy amplitude for the mass point subjected to an angular barrier near the surface of a sphere in $D = 4$ dimension's by

$$\begin{aligned}
G(u'', u'; E_{RM'}) &= \frac{1}{\sqrt{\sin \theta'' \sin \theta'}} G(\theta'', \theta'; E_{PT'}) \\
&= -\frac{i}{\hbar} \frac{\Gamma(M_1 - L_E) \Gamma(L_E + M_1 + 1)}{\Gamma(M_1 + M_2 + 1) \Gamma(M_1 - M_2 + 1)} \\
&\quad \times \left(\frac{1 + \tanh u'}{2} \right)^{(M_1 - M_2)/2} \left(\frac{1 - \tanh u'}{2} \right)^{(M_1 + M_2)/2} \\
&\quad \times \left(\frac{1 - \tanh u''}{2} \right)^{(M_1 + M_2)/2} \left(\frac{1 + \tanh u''}{2} \right)^{(M_1 - M_2)/2} \\
&\quad \times F \left(M_1 - L_E, L_E + M_1 + 1; M_1 - M_2 + 1; \frac{1 + \tanh u'}{2} \right) \\
&\quad \times F \left(M_1 - L_E, L_E + M_1 + 1; M_1 + M_2 + 1; \frac{1 - \tanh u''}{2} \right), \\
\end{aligned} \tag{79}$$

with $\tanh u = -\cos \theta$, $\theta \in (0, \pi)$, $u \in]-\infty, +\infty[$ and $u'' \succ u'$. Here the mass point is taken equal to unity. In addition, we set

$$\begin{cases} L_E = -\frac{1}{2} + \left(\frac{1}{16} + \frac{2E_{PT'}}{\hbar} \right)^{\frac{1}{2}}, \\ E_{PT'} = \frac{\hbar^2}{8} \left(\frac{E^2}{\hbar^2 \omega^2} + 4\beta - \alpha - \frac{5}{4} \right), \end{cases} \tag{80}$$

and if we choose

$$\begin{cases} M_1 = \frac{1}{2} \left(\sqrt{N^2 - 2\beta - \alpha + \frac{11}{4}} + l + \frac{1}{2} \right), \\ M_2 = \frac{1}{2} \left(\sqrt{N^2 - 2\beta - \alpha + \frac{11}{4}} - l - \frac{1}{2} \right), \end{cases} \tag{81}$$

the boundary conditions for the wave functions appearing in (79) will be satisfied.

The equivalence between the relativistic harmonic oscillator interaction in $(3 + 1)$ Minkowski space-time and a free relativistic particle in CAdS is characterized by the following restriction on the parameters α and β :

$$\alpha = 8\xi, \quad \beta = 2\xi + \frac{1}{4}. \quad (82)$$

Inserting (79) into (76), we get, for the radial Green's function, the closed form:

$$\begin{aligned} G_l(r'', r') = & -\frac{\Gamma(M_1 - L_E)\Gamma(L_E + M_1 + 1)}{4\hbar^2\omega c\Gamma(M_1 + M_2 + 1)\Gamma(M_1 - M_2 + 1)} \\ & \times (\Lambda(r'')\Lambda(r') \cosh u'' \cosh u')^{-\frac{1}{2}} \\ & \times \left(\frac{1 + \tanh u'}{2}\right)^{(M_1 - M_2)/2} \left(\frac{1 - \tanh u'}{2}\right)^{(M_1 + M_2)/2} \\ & \times \left(\frac{1 - \tanh u''}{2}\right)^{(M_1 + M_2)/2} \left(\frac{1 + \tanh u''}{2}\right)^{(M_1 - M_2)/2} \\ & \times F\left(M_1 - L_E, L_E + M_1 + 1; M_1 - M_2 + 1; \frac{1 + \tanh u'}{2}\right) \\ & \times F\left(M_1 - L_E, L_E + M_1 + 1; M_1 + M_2 + 1; \frac{1 - \tanh u''}{2}\right). \end{aligned} \quad (83)$$

The poles of (83) are all contained in the first Γ function in the numerator,

$$M_1 - L_E = -n_r. \quad (84)$$

Converting this into energy by using Eqs. (80), (81) and (84) yields

$$E_{n_r, l} = \beta_{n_r, l} \hbar \omega, \quad (85)$$

with

$$\beta_{n_r, l} = 2n_r + l + \frac{1}{2}\sqrt{9 + 4N^2 - 48\xi} + \frac{3}{2}, \quad (86)$$

where n_r is the radial quantum number and l the angular momentum. Here the parameter ξ is subject to the condition $\xi \prec \frac{1}{48}(9 + 4N^2)$.

In the non-relativistic approximation

$$E_{n_r, l} \xrightarrow{c \rightarrow \infty} \left(2n_r + l + \frac{3}{2}\right) \hbar \omega + Mc^2, \quad (87)$$

where the first term represents the well-known energy spectrum of the three-dimensional non-relativistic harmonic oscillator.

As in the one-dimensional case, the radial wave functions can be found by approximation near the poles $M_1 - L_E \approx -n_r$:

$$\begin{aligned}\Gamma(M_1 - L_E) &\approx \frac{(-1)^{n_r}}{n_r!} \frac{1}{M_1 - L_E + n_r} \\ &= \frac{(-1)^{n_r+1}}{n_r!} \frac{4\hbar^2\omega^2\beta_{n_r,l}}{E^2 - E_{n_r,l}^2}.\end{aligned}\quad (88)$$

Using this behavior and taking into consideration the Gauss transformation formula (see Eq. (9.131.2) , p. 1043 in Ref.²⁹)

$$\begin{aligned}F(a, b, c; z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a, b, a+b-c+1; 1-z) + \\ &\quad \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} \\ &\quad \times F(c-a, c-b, c-a-b+1; 1-z),\end{aligned}\quad (89)$$

knowing that the second term of this latter is null because the Euler function $\Gamma(a)$ is infinite ($a = -n_r \leq 0$), we can write Eq. (83) as

$$G_l(r'', r') = \sum_{n_r=0}^{\infty} \frac{\Phi_{n_r,l}(r'')\Phi_{n_r,l}^*(r')}{E^2 - E_{n_r,l}^2}, \quad (90)$$

where

$$\begin{aligned}\Phi_{n_r,l}(r) &= \left[\frac{2\beta_{n_r,l}\Gamma(\beta_{n_r,l} - n_r)\Gamma(n_r + l + \frac{3}{2})}{n_r!\Gamma(\beta_{n_r,l} - n_r - \frac{1}{2})} \right]^{\frac{1}{2}} \frac{1}{\Gamma(l + \frac{3}{2})} \left(\frac{\omega}{c}\right)^{l+\frac{3}{2}} r^l \\ &\quad \left(1 + \frac{\omega^2}{c^2}r^2\right)^{n_r-\beta_{n_r,l}/2} F\left(-n_r, \beta_{n_r,l} - n_r; l + \frac{3}{2}; \frac{\frac{\omega^2}{c^2}r^2}{1 + \frac{\omega^2}{c^2}r^2}\right)\end{aligned}\quad (91)$$

are the radial wave functions.

By substituting (see Eq. (9.131.1), p. 1043 in Ref.²⁹)

$$F(\alpha, \beta; \gamma; z) = (1 - z)^{-\alpha} F(\alpha, \gamma - \beta; \gamma; \frac{z}{z - 1}) \quad (92)$$

into (91) and using the connecting formula (see Eq. (8.962.1), p. 1036 in Ref.²⁹)

$$P_n^{(\alpha, \beta)}(x) = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)} F\left(-n, n + \alpha + \beta + 1; \alpha + 1; \frac{1 - x}{2}\right), \quad (93)$$

we can also express (91) in the form

$$\begin{aligned} \Phi_{n_r, l}(r) &= \left[\frac{2\beta_{n_r, l} n_r! \Gamma(\beta_{n_r, l} - n_r)}{\Gamma(n_r + \gamma + 1) \Gamma(n_r + l + \frac{3}{2})} \right]^{\frac{1}{2}} \left(\frac{\omega}{c} \right)^{l + \frac{3}{2}} r^l \\ &\times \left(1 + \frac{\omega^2}{c^2} r^2 \right)^{-\frac{1}{2}\beta_{n_r, l}} P_{n_r}^{(l + \frac{1}{2}, -\beta_{n_r, l})} \left(1 + 2\frac{\omega^2}{c^2} r^2 \right). \end{aligned} \quad (94)$$

By using the following limiting relations:

$$\left\{ \begin{array}{l} \lim_{\gamma \rightarrow \infty} \Gamma(n + \gamma + 1) = \lim_{\gamma \rightarrow \infty} \gamma^{n+1} \Gamma(\gamma), \\ \lim_{\gamma \rightarrow \infty} \left(1 + \frac{\omega^2}{c^2} r^2 \right)^{-\frac{1}{2}(l + \gamma + \frac{3}{2})} = e^{-\frac{M\omega}{2\hbar} r^2}, \\ \lim_{\gamma \rightarrow \infty} F\left(\alpha, \gamma; \nu; \frac{z}{\gamma}\right) = F(\alpha, \nu; z), \end{array} \right. \quad (95)$$

we obtain the well-known radial wave functions of the non-relativistic harmonic oscillator

$$\begin{aligned} \Phi_{n_r, l}(r) &= \left[\frac{2\Gamma(n_r + l + \frac{3}{2})}{n_r!} \right]^{\frac{1}{2}} \left(\frac{M\omega}{\hbar} \right)^{\frac{1}{2}(l + \frac{3}{2})} \frac{r^l}{\Gamma(l + \frac{3}{2})} \exp\left[-\frac{M\omega}{2\hbar} r^2\right] \\ &\times F\left(-n_r, l + \frac{3}{2}; \frac{M\omega}{\hbar} r^2\right) \end{aligned} \quad (96)$$

IV. Conclusion

In this paper we have dealt special relativistic harmonic oscillators in $(1+1)$ - and $(3+1)$ -dimensional Minkowski space-time modeled by a free relativistic particle in the universal covering space-time of the anti-de Sitter space-time. The explicit path integral solution presented above provides a valuable alternative way to the one obtained through the Klein-Gordon equation. After formulating the problem in terms of symmetric and general Rosen-Morse potentials for the one- and three-dimensional relativistic oscillators respectively and by imposing a restriction on the parameters α and β in such a way the system under consideration is equivalent to a free relativistic particle in CAdS, the Green's functions are obtained in a closed form. The energy spectrum and the properly normalized wave functions are extracted from the poles and the residues at the poles of the Green's function respectively. In the flat-space limit ($R \rightarrow 0$) i.e., in the non-relativistic approximation ($c \rightarrow \infty$), the usual harmonic oscillator spectrum and the corresponding normalized wave functions are regained.

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